

New modification of Maheshwari method with optimal eighth order of convergence for solving nonlinear equations

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Abstract

In this paper, we present a family of three-point with eight-order convergence methods for finding the simple roots of nonlinear equations by suitable approximations and weight function based on Maheshwari method. Per iteration this method requires three evaluations of the function and one evaluation of its first derivative. This class of methods has the efficiency index equal to $8^{\frac{1}{4}} \approx 1.682$. We describe the analysis of the proposed methods along with numerical experiments including comparison with existing methods.

Keywords: Multi-point iterative methods; Simple root; Maheshwari method; Kung and Traub's conjecture.

1 Introduction

Finding roots of nonlinear functions $f(x) = 0$ by using iterative methods is a classical problem which has interesting applications in different branches of science, in particular in physics and engineering. Therefore, several numerical methods for approximating simple roots of nonlinear equations have been developed and analyzed by using various techniques based on iterative methods in the recent years. The second order Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is one of the best-known iterative methods for finding approximate roots and it requires two evaluations for each iteration step, one evaluation of f and one of f' [12, 16].

Kung and Traub [8] conjectured that no multi-point method without memory with n evaluations could have a convergence order larger than 2^{n-1} . A multi-point method with convergence order 2^{n-1} is called optimal. The efficiency index provides a measure of the balance between those quantities, according to the formula $p^{1/n}$, where p is the convergence order of the method and n is the number of function evaluations per iteration.

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Some well known two-point methods without memory are described by e.g. Jarratt [6], King [7], Maheshwari [11] and Ostrowski [12]. Using inverse interpolation, Kung and Traub [8] constructed two general optimal classes without memory. Since then, there have been many attempts to construct optimal multi-point methods, utilizing e.g. weight functions [1], [2], [4], [9], [10], [13], [14], [15], [17] and [18].

In this paper, we construct a new class of optimal eight order of convergence based on Maheshwari method. This paper is organized as follows: Section 2 is devoted to the construction and convergence analysis of the new class. In Section 3, the new methods are compared with a closest competitor in a series of numerical examples, and Section 4 contains a short conclusion.

2 Description of the method and convergence analysis

2.1 Three-point method of optimal order of convergence

In this section we propose a new optimal three-point method based on Maheshwari method [11] for solving nonlinear equations. The Maheshwari method is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \end{cases} \quad (n = 0, 1, \dots), \quad (2.1)$$

where x_0 is an initial approximation of x^* . The convergence order of (2.1) is four with three functional evaluations per iteration such that this method is optimal. We intend to increase the order of convergence of method (2.1) by an additional Newton step. So we obtain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \end{cases} \quad (2.2)$$

Method (2.2) uses five function evaluations with order of convergence eight. Therefore, this method is not optimal. In order to decrease the number of function evaluations, we approximate $f'(z_n)$ by an expression based on $f(x_n)$, $f(y_n)$, $f(z_n)$ and $f'(x_n)$. Therefore

$$f'(z_n) \approx \frac{f'(x_n)}{F(x_n, y_n, z_n)H(s_n)}, \quad (2.3)$$

where

$$F(x_n, y_n, z_n) = \left(\frac{f^3(y_n)(f(x_n) - 10f(y_n)) + 4f^2(x_n)(f^2(y_n) + f(x_n)f(y_n))}{f(x_n)(2f(x_n) - f(y_n))^2(f(y_n) - f(z_n))} \right), \quad (2.4)$$

and $s_n = \frac{f(z_n)}{f(x_n)}$.

We have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} F(x_n, y_n, z_n) H(s_n), \end{cases} \quad (2.5)$$

where $F(x_n, y_n, z_n)$ and s_n are defined as above.

2.2 Convergence analysis

In the following theorem we provide sufficient conditions on the weight function $H(s_n)$ which imply that method (2.5) has convergence order eight.

Theorem 2.2.1. *Assume that function $f : D \rightarrow \mathbb{R}$ is eight times continuously differentiable on an interval $D \subset \mathbb{R}$ and has a simple zero $x^* \in D$. Moreover, H is one time continuously differentiable. If the initial approximation x_0 is sufficiently close to x^* then the class defined by (2.5) converges to x^* and the order of convergence is eight under the conditions*

$$H(0) = 1, \quad H'(0) = 2,$$

with the error equation

$$e_{n+1} = \left(\frac{1}{2}c_2^2(4c_2^2 - c_3)(c_2^3 - 8c_2c_3 + 2c_4) \right) e_n^8 + O(e_n^9).$$

Proof. Let $e_n := x_n - x^*$, $e_{n,y} := y_n - x^*$, $e_{n,z} := z_n - x^*$ and $c_n := \frac{f^{(n)}(x^*)}{n!f'(x^*)}$ for $n \in \mathbb{N}$. Using the fact that $f(x^*) = 0$, Taylor expansion of f at x^* yields

$$f(x_n) = f'(x^*) (e_n + c_2e_n^2 + c_3e_n^3 + \cdots + c_8e_n^8) + O(e_n^9), \quad (2.6)$$

and

$$f'(x_n) = f'(x^*) (1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots + 9c_9e_n^8) + O(e_n^9). \quad (2.7)$$

Therefore

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &\quad + (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 \\ &\quad + (-16c_2^5 + 52c_2^3c_3 - 28c_2^2c_4 + 17c_3c_4 - c_2(33c_3^2 - 13c_5))e_n^6 + O(e_n^7), \end{aligned}$$

and hence

$$\begin{aligned} e_{n,y} = y_n - x^* &= c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad + (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 \\ &\quad + (16c_2^5 - 52c_2^3c_3 + 28c_2^2c_4 - 17c_3c_4 + c_2(33c_3^2 - 13c_5))e_n^6 + O(e_n^7). \end{aligned}$$

For $f(y_n)$ we also have

$$f(y_n) = f'(x^*) (e_{n,y} + c_2e_{n,y}^2 + c_3e_{n,y}^3 + \cdots + c_8e_{n,y}^8) + O(e_{n,y}^9). \quad (2.8)$$

Therefore by substituting (2.6), (2.7) and (2.8) into (2.2), we get

$$\begin{aligned} e_{n,z} = z_n - x^* &= (4c_2^3 - c_2c_3)e_n^4 + (-27c_2^4 + 26c_2^2c_3 - 2c_3^2 - 2c_2c_4)e_n^5 \\ &\quad + (120c_2^5 - 196c_2^3c_3 + 39c_2^2c_4 - 7c_3c_4 + c_2(54c_3^2 - 3c_5))e_n^6 + O(e_n^7). \end{aligned}$$

For $f(z_n)$ we also get

$$f(z_n) = f'(x^*) (e_{n,z} + c_2e_{n,z}^2 + c_3e_{n,z}^3 + \cdots + c_8e_{n,z}^8) + O(e_{n,z}^9). \quad (2.9)$$

From (2.6), (2.8) and (2.9) we obtain

$$\begin{aligned} F(x_n, y_n, z_n) = & 1 + 2c_2e_n + 3c_3e_n^2 + (-8c_2^3 + 2c_2c_3 + 4c_4)e_n^3 + \left(\frac{83}{2}c_2^4 - 45c_2^2c_3 + 4c_3^2 + 3c_2c_4 \right. \\ & + 5c_5)e_n^4 + \left(\frac{7167}{8}c_2^6 + 1731c_2^4c_3 - 56c_3^3 + 429c_2^3c_4 - 245c_2c_3c_4 + 9c_4^2 + c_2^2(815c_3^2 \right. \\ & \left. - 84c_5) + 16c_3c_5)e_n^6 + O(e_n^7). \end{aligned} \quad (2.10)$$

From (2.6) and (2.9) we have

$$\begin{aligned} s_n = & (4c_2^3 - c_2c_3)e_n^3 + (-31c_2^4 + 27c_2^2c_3 - 2c_3^2 - 2c_2c_4)e_n^4 + (151c_2^5 \\ & - 227c_2^3c_3 + 41c_2^2c_4 - 7c_3c_4 + c_2(57c_2^3 - 3c_5))e_n^5 + (592c_2^6 + 1266c_2^4c_3 \\ & + 38c_3^3 - 325c_2^3c_4 + 170c_2c_3c_4 - 6c_4^2 - 10c_3c_5 + c_2^2(-608c_3^2 + 55c_5))e_n^6 + O(e_n^7). \end{aligned} \quad (2.11)$$

Expanding H at 0 yields

$$H(s_n) = H(0) + H'(0)s_n + O(s_n^2). \quad (2.12)$$

Substituting (2.6)-(2.12) into (2.5) we obtain

$$e_{n+1} = x_{n+1} - x^* = R_4e_n^4 + R_5e_n^5 + R_6e_n^6 + R_7e_n^7 + R_8e_n^8 + O(e_n^9),$$

where

$$\begin{aligned} R_4 &= -c_2(4c_2^2 - c_3)(-1 + H(0)), \\ R_5 &= 0, \\ R_6 &= 0, \\ R_7 &= -c_2^2(-4c_2^2 + c_3)^2(-2 + H'(0)). \end{aligned}$$

By setting $R_4 = R_7 = 0$ and $R_8 \neq 0$ the convergence order becomes eight. Obviously

$$\begin{aligned} H(0) = 1 & \Rightarrow R_4 = 0, \\ H'(0) = 2 & \Rightarrow R_7 = 0, \end{aligned}$$

consequently the error equation becomes

$$e_{n+1} = \left(\frac{1}{2}c_2^2(4c_2^2 - c_3)(c_2^3 - 8c_2c_3 + 2c_4)\right)e_n^8 + O(e_n^9),$$

which finishes the proof of the theorem. \square

In what follows we give some concrete explicit representations of (2.5) by choosing different weight function satisfying the provided condition for the weight function $H(s_n)$ in Theorem 2.2.1.

Method 1. Choose the weight function H as:

$$H(s) = 1 + 2s. \quad (2.13)$$

The function H in (2.13) satisfies the assumptions of Theorem 2.2.1 and we get

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left(\frac{f(x_n) + 2f(z_n)}{f(x_n)} \right) F(x_n, y_n, z_n), \end{cases} \quad (2.14)$$

where $F(x_n, y_n, z_n)$ is evaluated by (2.4).

Method 2. Choose the weight function H as:

$$H(s) = \frac{1 + 4s}{1 + 2s}. \quad (2.15)$$

The function H in (2.15) satisfies the assumptions of Theorem 2.2.1 and we obtain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left(\frac{f(x_n) + 4f(z_n)}{f(x_n) + 2f(z_n)} \right) F(x_n, y_n, z_n), \end{cases} \quad (2.16)$$

where $F(x_n, y_n, z_n)$ is evaluated by (2.4).

Method 3. Choose the weight function H as:

$$H(s) = \frac{1}{1 - 2s}. \quad (2.17)$$

The function H in (2.17) satisfies the assumptions of Theorem 2.2.1 and we get

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left(\frac{f(x_n)}{f(x_n) - 2f(z_n)} \right) F(x_n, y_n, z_n), \end{cases} \quad (2.18)$$

where $F(x_n, y_n, z_n)$ is evaluated by (2.4).

We apply the new methods (2.14), (2.16) and (2.18) to several benchmark examples and compare them with existing three-point methods which have the same order of convergence and the same computational efficiency index equal to $\sqrt[\theta]{r} = 1.682$ for the convergence order $r = 8$ which is optimal for $\theta = 4$ function evaluations per iteration [12, 16].

3 Numerical performance

In this section we test and compare our proposed methods with some existing methods. We compare methods (2.14), (2.16) and (2.18) with the following related three-point methods.

Bi, Ren and Wu method. The method by Bi et al. [1] is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)} H(t), \end{cases} \quad (3.1)$$

where $\beta = -\frac{1}{2}$ and weight function

$$H(t) = \frac{1}{(1 - \alpha t)^2}, \quad \alpha = 1. \quad (3.2)$$

Wang and Liu method. The method by Wang and Liu [18] is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{f(x_n)}{f'(x_n)} G(t), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} (H(t) + V(t)W(s)), \end{cases} \quad (3.3)$$

with weight functions

$$G(t) = \frac{1-t}{1-2t}, \quad H(t) = \frac{5-2t+t^2}{5-12t}, \quad V(t) = 1+4t, \quad W(s) = s. \quad (3.4)$$

Sharma and Sharma method. The Sharma and Sharma method [15] is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} = z_n - \frac{f[x_n, y_n]f(z_n)}{f[x_n, z_n]f[y_n, z_n]} W(t), \end{cases} \quad (3.5)$$

with weight function

$$W(t) = 1 + \frac{t}{1 + \alpha t}, \quad \alpha = 1. \quad (3.6)$$

The three point method (2.5) is tested on a number of nonlinear equations. To obtain a high accuracy and avoid the loss of significant digits, we employed multi-precision arithmetic with 7000 significant decimal digits in the programming package of Mathematica 8.

In order to test our proposed methods (2.14), (2.16) and (2.18) and also compare them with the methods (3.1), (3.3) and (3.5) we choose the initial value x_0 using the **Mathematica** command **FindRoot** [5, pp. 158–160] and compute the error and the approximated computational order of convergence, (ACOC) by the formula [3]

$$\text{ACOC} \approx \frac{\ln |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}.$$

M	W-F	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	$ x_4 - x^* $	ACOC
(2.14)	(2.13)	$0.937e-8$	$0.655e-63$	$0.374e-504$	$0.865e-1913$	8.0000
(2.16)	(2.15)	$0.568e-4$	$0.145e-30$	$0.259e-243$	$0.272e-1945$	8.0000
(2.18)	(2.17)	$0.755e-4$	$0.141e-29$	$0.206e-235$	$0.423e-1882$	8.0000
(3.1)	(3.2)	$0.720e-4$	$0.584e-30$	$0.110e-238$	$0.175e-1908$	8.0000
(3.3)	(3.4)	$0.278e-3$	$0.779e-26$	$0.296e-206$	$0.128e-1649$	8.0000
(3.5)	(3.6)	$0.753e-4$	$0.619e-31$	$0.128e-247$	$0.453e-1981$	8.0000

Table 1: Comparison for $f(x) = \ln(1+x^2) + e^{x^2-3x} \sin(x)$, $x^* = 0$, $x_0 = 0.35$, for different methods (M) and weight functions (W-F).

M	W-F	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	$ x_4 - x^* $	ACOC
(2.14)	(2.13)	$0.444e - 11$	$0.399e - 94$	$0.170e - 758$	$0.189e - 6073$	8.0000
(2.16)	(2.15)	$0.445e - 11$	$0.404e - 94$	$0.187e - 758$	$0.394e - 6073$	8.0000
(2.18)	(2.17)	$0.443e - 11$	$0.395e - 94$	$0.155e - 758$	$0.902e - 6077$	8.0000
(3.1)	(3.2)	$0.423e - 12$	$0.134e - 114$	$0.445e - 1037$	$0.211e - 9339$	9.0000
(3.3)	(3.4)	$0.225e - 11$	$0.629e - 96$	$0.265e - 773$	$0.264e - 6192$	8.0000
(3.5)	(3.6)	$0.172e - 11$	$0.581e - 98$	$0.984e - 790$	$0.663e - 6324$	8.0000

Table 2: Comparison for $f(x) = \ln(1 - x + x^2) + 4 \sin(1 - x)$, $x^* = 1$, $x_0 = 1.1$, for different methods (M) and weight functions (W-F).

M	W-F	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	$ x_4 - x^* $	ACOC
(2.14)	(2.13)	$0.783e - 8$	$0.648e - 64$	$0.142e - 512$	$0.765e - 4102$	8.0000
(2.16)	(2.15)	$0.749e - 8$	$0.656e - 64$	$0.855e - 514$	$0.132e - 4111$	8.0000
(2.18)	(2.17)	$0.816e - 8$	$0.908e - 64$	$0.212e - 511$	$0.187e - 4092$	8.0000
(3.1)	(3.2)	$0.673e - 8$	$0.113e - 64$	$0.726e - 519$	$0.208e - 4152$	8.0000
(3.3)	(3.4)	$0.997e - 10$	$0.751e - 80$	$0.782e - 641$	$0.107e - 5128$	8.0000
(3.5)	(3.6)	$0.642e - 10$	$0.101e - 81$	$0.389e - 656$	$0.184e - 5251$	8.0000

Table 3: Comparison for $f(x) = x^4 + \sin(\frac{\pi}{x^2}) - 5$, $x^* = \sqrt{2}$, $x_0 = 1.5$, for different methods (M) and weight functions (W-F).

M	W-F	$ x_1 - x^* $	$ x_2 - x^* $	$ x_3 - x^* $	$ x_4 - x^* $	ACOC
(2.14)	(2.13)	$0.119e - 3$	$0.253e - 26$	$0.106e - 207$	$0.992e - 1659$	8.0000
(2.16)	(2.15)	$0.143e - 3$	$0.109e - 25$	$0.124e - 202$	$0.353e - 1618$	8.0000
(2.18)	(2.17)	$0.916e - 4$	$0.307e - 27$	$0.493e - 215$	$0.221e - 1717$	8.0000
(3.1)	(3.2)	$0.183e - 4$	$0.319e - 33$	$0.278e - 263$	$0.920e - 2104$	8.0000
(3.3)	(3.4)	$0.612e - 4$	$0.111e - 28$	$0.134e - 220$	$0.588e - 1810$	7.9999
(3.5)	(3.6)	$0.239e - 4$	$0.138e - 32$	$0.170e - 258$	$0.938e - 2066$	8.0000

Table 4: Comparison for $f(x) = (x - 2)(x^{10} + x + 1)e^{-x-1}$, $x^* = 2$, $x_0 = 2.1$, for different methods (M) and weight functions (W-F).

In Tables 1, 2, 3 and 4, the proposed methods (2.14), (2.16) and (2.18) with the methods (3.1), (3.3) and (3.5) have been tested on different nonlinear equations. It is clear that these methods are in accordance with the developed theory.

4 Conclusion

We presented a new optimal class of three-point methods without memory for approximating a simple root of a given nonlinear equation. Our proposed methods use five function evaluations for each iteration. Therefore they support Kung and Traub's conjecture. Numerical examples show that our methods work and can compete with other methods in the same class.

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